

COHERENCE WITHOUT UNIQUE NORMAL FORMS

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ABSTRACT. Coherence theorems for covariant structures carried by a category have traditionally relied on the underlying term rewriting system of the structure being terminating and confluent. While this holds in a variety of cases, it is not a feature that is inherent to the coherence problem itself. This is demonstrated by the theory of iterated monoidal categories, which model iterated loop spaces and have a coherence theorem but fail to be confluent. We develop a framework for expressing coherence problems in terms of term rewriting systems equipped with a two dimensional congruence. Within this framework we provide general solutions to two related coherence theorems: Determining whether there is a decision procedure for the commutativity of diagrams in the resulting structure and determining sufficient conditions ensuring that “all diagrams commute”. The resulting coherence theorems rely on neither the termination nor the confluence of the underlying rewriting system. We apply the theory to iterated monoidal categories and obtain a new, conceptual proof of their coherence theorem.

1. INTRODUCTION

Coherence theorems are a mechanism for ensuring that an extra structure carried by a category is not too wildly behaved. This typically takes the form of an assurance that a certain large class of diagrams always commutes. In the most favourable situation, one proves that any diagram built solely out of the structuring functors and natural transformations is guaranteed to commute. This was the case in the earliest coherence results of Mac Lane for monoidal and symmetric monoidal categories [11].

A close examination of Mac Lane’s proof reveals a connection between covariant structures carried by categories and term rewriting theory. In particular, the proof mainly revolves around elucidating the fact that a free monoidal structure on a discrete category, considered as a term rewriting system, is terminating and confluent. “Termination” means that there are no infinite chains of non-identity morphisms, while “confluence” is the property that every span may be completed into a square (see Figure 1).

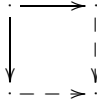


FIGURE 1. Confluence

Confluence and termination together conspire to ensure that a term rewriting system has unique normal forms. That is, not only is every chain of morphisms

finite, but every sequence of morphisms beginning from an object ends at a point that depends only on the starting object. This seemingly strong property is present in a very large array of structures and has, for instance, been exploited by Laplaza to derive coherence theorems for directed associativity [9] and for distributive categories [10].

Unfortunately, it is simply not the case that every coherent covariant structure has unique normal forms. For instance, the structure consisting of a unary functor F and the single natural transformation $F(X) \rightarrow F(F(X))$ is non-terminating, but easily seen to be coherent. A more spectacular counterexample to the hope that coherent structures have unique normal forms is provided by the theory of iterated monoidal categories [2], which arise as a categorical model of iterated loop spaces and fail to be confluent.

We are now faced with the problem of determining sufficient conditions for coherence in terms of the underlying rewriting system of a structure that do not rely on either termination or confluence. This very quickly leads one to consider two further coherence questions: If there are diagrams that do not commute, then is there at least a decision procedure that determines whether a given diagram commutes? Is it at least true that for any finite collection of functors and natural transformations, there is always a finite set of diagrams whose commutativity implies the commutativity of all diagrams built from this structure?

This paper sets out to solve the various coherence questions by vigorously pursuing the idea that two morphisms with the same source and target in a free covariant structure on a discrete category commute precisely when they admit a planar subdivision such that each face is an instance of naturality, or of functoriality or of one of the axioms. The guiding intuition behind this approach is that a span that cannot be completed into a square can never appear in such a subdivision.

We begin in Section 2 by developing a framework for viewing a two dimensional structure on a category as a term rewriting system modulo a two dimensional congruence. In Section 3, we resolve the problem of determining sufficient conditions for the existence of a decision procedure for the commutativity of diagrams. We call this problem the “Lambek coherence problem”, since it is inspired by Lambek’s paper on closed categories and deductive systems [8]. In Section 4, we determine sufficient conditions for all diagrams to commute, a problem that we call the “Mac Lane coherence problem”. As an immediate application, we construct an example of a structure that has no finite basis for Mac Lane coherence but is otherwise well behaved. Finally, in Section 5, we apply the theory to iterated monoidal categories and obtain a new and conceptual proof of their coherence theorem.

2. 2-STRUCTURES

The purpose of this section is to describe a two-dimensional covariant structure on a category as a certain type of term rewriting system. At the onset, we are presented with certain basic functors and natural transformations, together with an equational theory on the absolutely free term algebra generated by the functors, as well as an equational theory on the absolutely free reduction system generated by the natural transformations. The idea of viewing such a system as a term rewriting system can be found, for instance, in Meseguer’s Rewriting Logic [12]. An important point to note is that Rewriting Logic does not allow any additional equations on reductions, beyond those required to ensure naturality and functoriality. In other

words, it does not provide a facility for specifying coherence conditions. We begin by describing the first layer of structure.

Definition 2.1. *Given a graded set of function symbols $\mathcal{F} := \sum_n \mathcal{F}_n$ and a set X , the absolutely free term algebra generated by \mathcal{F} on X is denoted by $\text{Term}_X(\mathcal{F})$.*

The next layer of structure adds an equational theory to $\text{Term}_X(\mathcal{F})$:

Definition 2.2. *Given a graded set of function symbols \mathcal{F} , a set X and a set of equations $\theta_{\mathcal{F}}$ on $\text{Term}_X(\mathcal{F})$, we denote by $\text{Term}_X(\mathcal{F}, \theta_{\mathcal{F}})$ the quotient of $\text{Term}_X(\mathcal{F})$ by the smallest congruence generated by $\theta_{\mathcal{F}}$. We write $[t]$ for the image of a term t under the homomorphism $\text{Term}_X(\mathcal{F}) \rightarrow \text{Term}_X(\mathcal{F}, \theta_{\mathcal{F}})$.*

The next layer of structure adds some reduction rules between congruence classes of $\text{Term}_X(\mathcal{F}, \theta_{\mathcal{F}})$.

Definition 2.3. *A labelled term rewriting theory is a tuple $\langle X, \mathcal{F}, \theta_{\mathcal{F}}, \mathcal{L}, \mathcal{T} \rangle$, where X is a countably infinite set of variables, \mathcal{F} is a graded set of function symbols, $\theta_{\mathcal{F}}$ is a system of $\text{Term}_X(\mathcal{F})$ -equations, \mathcal{L} is a set of labels and \mathcal{T} is a subset of $\mathcal{L} \times (\text{Term}_X(\mathcal{F}, \theta_{\mathcal{F}}))^2$ satisfying the following consistency conditions:*

If (α, s_1, t_1) and (α, s_2, t_2) are in \mathcal{T} then $s_1 = s_2$ and $t_1 = t_2$.

If $(\alpha, s, t) \in \mathcal{T}$, we write $\alpha : \ell \rightarrow r$. A member of \mathcal{T} is called a labelled reduction rule.

Given a labelled term rewriting theory $\langle X, \mathcal{F}, \theta_{\mathcal{F}}, \mathcal{L}, \mathcal{T} \rangle$, the particular choice of X and \mathcal{L} is irrelevant. What is important is simply that there are sufficient variables and labels. Accordingly, we shall henceforth suppress explicit mention of the variables and labels and write $\langle \mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T} \rangle$ for a labelled term rewriting theory. A labelled term rewriting theory embodies the basic reductions that are to generate all others. The next step is to obtain an analogue of the absolutely free term algebra for this higher dimensional layer of structure. This is achieved by the following definition, we there notation \bar{x}^n is an abbreviation for x_1, \dots, x_n and $F(\bar{s}^n / \bar{x}^n)$ denotes the uniform substitution of the free variables \bar{x}^n by \bar{s}^n .

Definition 2.4. *Given a labelled term rewriting theory $\mathcal{R} := \langle \mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T} \rangle$ and a category \mathcal{C} , the set of reductions generated by \mathcal{R} is denoted $\text{Term}_{\mathcal{C}}(\mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T})$ and is generated inductively by the following rules:*

$$\begin{array}{c}
 \frac{}{[f] : [s] \rightarrow [t]} \quad (\text{Inheritance}) \\
 \\
 \frac{\varphi_1 : [s_1] \rightarrow [t_1] \quad \dots \quad \varphi_n : [s_n] \rightarrow [t_n]}{F(\varphi_1, \dots, \varphi_n) : [F(s_1, \dots, s_n)] \rightarrow [F(t_1, \dots, t_n)]} \quad (\text{Structure}) \\
 \\
 \frac{\alpha : [F(\bar{x}^n)] \rightarrow [G(\bar{x}^n)] \quad (\varphi_i : [s_i] \rightarrow [t_i])_{i=1}^n}{\tau(\varphi_1, \dots, \varphi_n) : [F(\bar{s}^n / \bar{x}^n)] \rightarrow [G(\bar{t}^n / \bar{x}^n)]} \quad (\text{Replacement}) \\
 \\
 \frac{\varphi : [s] \rightarrow [u] \quad \psi : [u] \rightarrow [t]}{(\varphi \cdot \psi) : [s] \rightarrow [t]} \quad (\text{Transitivity})
 \end{array}$$

In the (Inheritance) rule, $f : s \rightarrow t$ is in $\text{Mor}(\mathcal{C})$. In the (Structure) rule, F is a function symbol of rank n . In the (Replacement) rule.

Example 2.5. *Let \mathcal{C} be the discrete category generated by the set $\{A, B, C, D\}$. Consider the term rewriting theory with a single binary function symbol \otimes , an empty*

equational theory on terms and the single reduction rule:

$$\alpha(x, y, z) : x \otimes (y \otimes z) \rightarrow (x \otimes y) \otimes z$$

A derivation of $A \otimes (B \otimes (C \otimes D)) \rightarrow (A \otimes B) \otimes (C \otimes D)$ in this system is given by:

$$\frac{\frac{1_A : A \rightarrow A}{1_A \otimes \alpha(1_B, 1_C, 1_D) : A \otimes (B \otimes (C \otimes D)) \rightarrow (A \otimes B) \otimes (C \otimes D)}{\frac{\frac{1_B : B \rightarrow B \quad 1_C : C \rightarrow C \quad 1_D : D \rightarrow D}{\alpha(1_B, 1_C, 1_D) : B \otimes (C \otimes D) \rightarrow (B \otimes C) \otimes D}}{1_A \otimes \alpha(1_B, 1_C, 1_D) : A \otimes (B \otimes (C \otimes D)) \rightarrow (A \otimes B) \otimes (C \otimes D)}}$$

The consistency condition in Definition 2.3 easily yields the following lemma, which asserts that we may equate reductions with their labels.

Lemma 2.6. *Let \mathcal{C} be a category and $\langle \mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T} \rangle$ be a labelled term rewriting theory. Then:*

- (1) *If $\alpha : s \rightarrow t$ and $\alpha : s' \rightarrow t'$ are in $\text{Term}_{\mathcal{C}}(\mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T})$, then $s = s'$ and $t = t'$.*
- (2) *For $t \in \text{Term}_{\text{Ob}(\mathcal{C})}(\mathcal{F}, \theta_{\mathcal{F}})$, there is a unique identity reduction $1_t : [t] \rightarrow [t]$ in $\text{Term}_{\mathcal{C}}(\mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T})$ given inductively by:*

$$1_t = \begin{cases} [1_t] & \text{if } t \in \text{Ob}(\mathcal{C}), \\ F(1_{t_1}, \dots, 1_{t_n}) & \text{if } t = F(t_1, \dots, t_n) \end{cases}$$

We now have the main ingredients for defining a covariant structure carried by a category. What remains is to ensure that the function symbols behave as functors, that the reduction rules behave as natural transformations and that we can stipulate coherence conditions.

Definition 2.7. *Let \mathcal{C} be a category. A covariant 2-structure on \mathcal{C} is a tuple $\langle \mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T}, \theta_{\mathcal{T}} \rangle$, where $\langle \mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T} \rangle$ is a labelled term rewriting theory and $\theta_{\mathcal{T}}$ is a set of equations on $\text{Term}_{\mathcal{C}}(\mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T})$ satisfying the following consistency condition:*

If $\varphi_1 = \varphi_2$ is in $\theta_{\mathcal{T}}$ and $\varphi_1 : s_1 \rightarrow t_1$ and $\varphi_2 : s_2 \rightarrow t_2$, then $s_1 = s_2$ and $t_1 = t_2$.

In other words, we can set two reductions to be equal only if their source and target match. We further stipulate that the following equations form a subset of $\theta_{\mathcal{T}}$.

$$\begin{aligned} 1_s \cdot \varphi &= \varphi & (\text{ID 1}) \\ \varphi \cdot 1_t &= \varphi & (\text{ID 2}) \\ \varphi \cdot (\psi \cdot \rho) &= (\varphi \cdot \psi) \cdot \rho & (\text{Assoc}) \\ F(\varphi_1, \dots, \varphi_n) \cdot F(\psi_1, \dots, \psi_n) &= F(\varphi_1 \cdot \psi_1, \dots, \varphi_n \cdot \psi_n) & (\text{Funct}) \\ \varphi(\varphi_1, \dots, \varphi_n) &= s(\varphi_1, \dots, \varphi_n) \cdot \varphi(t_1, \dots, t_n) & (\text{Nat 1}) \\ \varphi(\varphi_1, \dots, \varphi_n) &= \varphi(s_1, \dots, s_n) \cdot t(\varphi_1, \dots, \varphi_n) & (\text{Nat 2}) \end{aligned}$$

In the above, $\varphi : s \rightarrow t$ and $\varphi_i : s_i \rightarrow t_i$ are in $\text{Term}_{\mathcal{C}}(\mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T})$ and $F \in \mathcal{F}_n$.

Since the only structures we deal with in this paper are covariant, we shall henceforth take “2-structure” to mean “covariant 2-structure”. Our final task is to generate a congruence on reductions.

Definition 2.8. *If $\langle \mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T}, \theta_{\mathcal{T}} \rangle$ is a 2-structure on a category \mathcal{C} , then $\widehat{\theta_{\mathcal{T}}}$ denotes the smallest congruence generated by $\theta_{\mathcal{T}}$ on $\text{Term}_{\mathcal{C}}(\mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T})$. It is generated inductively by the following rules:*

$\frac{}{\varphi = \varphi}$	(Identity)	$\varphi \in \mathcal{T}$
$\frac{}{\varphi_1 = \varphi_2}$	(Inheritance)	$(\varphi_1, \varphi_2) \in \theta_{\mathcal{T}}$
$\frac{\varphi = \psi}{\psi = \varphi}$	(Symmetry)	
$\frac{\varphi_1 = \psi_1 \ \dots \ \varphi_n = \psi_n}{F(\varphi_1, \dots, \varphi_n) = F(\psi_1, \dots, \psi_n)}$	(Structure)	$F \in \mathcal{F}_n$
$\frac{\varphi_1 = \psi_1 \ \dots \ \varphi_n = \psi_n}{\tau(\varphi_1, \dots, \varphi_n) = \tau(\psi_1, \dots, \psi_n)}$	(Replacement)	$\tau \in \mathcal{T}_n$
$\frac{(\varphi_1 = \psi_1) : s \rightarrow u \quad (\varphi_2 = \psi_2) : u \rightarrow t}{(\varphi_1 \cdot \psi_1 = \varphi_2 \cdot \psi_2) : s \rightarrow t}$	(Transitivity)	

We are now in a position to define our main object of study.

Definition 2.9. *Given a 2-structure $\langle \mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T}, \theta_{\mathcal{T}} \rangle$ on a category \mathcal{C} , we use $\mathbb{F}_{\mathcal{C}}(\mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T}, \theta_{\mathcal{T}})$ to denote the quotient $\text{Term}_{\mathcal{C}}(\mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T}) / \widehat{\theta_{\mathcal{T}}}$.*

Our notion of a covariant 2-structure essentially recasts Kelly's definition of a fully covariant club [4] in the language of term rewriting theory. The construction of $\mathbb{F}_{\mathcal{C}}(s)$ parallels Kelly's construction of the functor part of an equational doctrine on **Cat** whose algebras are precisely the free s -algebras, relative to an appropriate notion of weak morphism between s -algebras. With this observation, we have the following theorem.

Theorem 2.10 (Kelly, [5]). *$\mathbb{F}_{\mathcal{C}}(s)$ is the initial s -algebra on \mathcal{C} .* □

Our main concern is to fully describe $\mathbb{F}_{\mathcal{C}}(s)$ in the case where \mathcal{C} is a discrete category in terms of the generators and relations in s . Moreover, we only wish to consider diagrams that are as general as possible. To this end, we formalise the notion that a reduction has “as many variables as possible”. We begin by defining the shape of a reduction.

Definition 2.11. *Let $\langle \mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T}, \theta_{\mathcal{T}} \rangle$ be a 2-structure on a category \mathcal{C} . The Shape of a reduction $\alpha \in \text{Term}_{\mathcal{C}}(\mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T})$ is defined recursively by the following:*

$$\text{Shape}(\alpha) = \begin{cases} \text{Shape}(\alpha_1) \cdot \text{Shape}(\alpha_2) & \text{if } \alpha = \alpha_1 \cdot \alpha_2 \\ \tau(\text{Shape}(\alpha_1), \dots, \text{Shape}(\alpha_n)) & \text{if } \alpha = \tau(\alpha_1, \dots, \alpha_n) \\ F(\text{Shape}(\alpha_1), \dots, \text{Shape}(\alpha_n)) & \text{if } \alpha = F(\alpha_1, \dots, \alpha_n) \\ \circ & \text{otherwise} \end{cases}$$

In the system from Example 2.5, we have:

$$\text{Shape}(\alpha(1_A, 1_B, 1_C)) = \text{Shape}(\alpha(1_A, 1_A, 1_A)) = \alpha(\circ, \circ, \circ)$$

We now need a precise definition of the variables present in a reduction.

Definition 2.12. Given a 2-structure $\langle \mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T}, \theta_{\mathcal{T}} \rangle$ on a category \mathcal{C} , the set of variables in a reduction $\alpha \in \text{Term}_{\mathcal{C}}(\mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T})$ is defined recursively as follows:

$$\text{Var}(\alpha) = \begin{cases} \text{Var}(\alpha_1) \cup \text{Var}(\alpha_2) & \text{if } \alpha = \alpha_1 \cdot \alpha_2 \\ \bigcup_{i=1}^n \text{Var}(\alpha_i) & \text{if } \alpha = \tau(\alpha_1, \dots, \alpha_n) \\ \bigcup_{i=1}^n \text{Var}(\alpha_i) & \text{if } \alpha = F(\alpha_1, \dots, \alpha_n) \\ \alpha & \text{otherwise} \end{cases}$$

Returning to Example 2.5, we find that $\text{Var}(\alpha(1_A, 1_B, 1_C)) = \{1_A, 1_B, 1_C\}$, whereas $\text{Var}(\alpha(1_A, 1_A, 1_A)) = \{1_A\}$. We can finally nail down what we mean when we say a reduction has the maximum possible number of variables.

Definition 2.13. Given a 2-structure $\langle \mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T}, \theta_{\mathcal{T}} \rangle$ on a category \mathcal{C} , a reduction $\alpha \in \text{Term}_{\mathcal{C}}(\mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T})$ is in general position if

$$|\text{Var}(\alpha)| = \max\{|\text{Var}(\tau)| : \tau \in \text{Term}_{\mathcal{C}}(\mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T}) \text{ and } \text{Shape}(\tau) = \text{Shape}(\alpha)\}.$$

Example 2.14. Consider the system from Example 2.5 augmented with the following reduction rule:

$$\beta(x) : x \otimes x \rightarrow x$$

Then,

$$\alpha(1_A, 1_A, 1_B) \cdot (\beta(1_A) \otimes 1_B) : A \otimes (A \otimes B) \rightarrow A \otimes B$$

is in general position, whereas

$$\alpha(1_A, 1_A, 1_B) : A \otimes (A \otimes B) \rightarrow (A \otimes A) \otimes B$$

is not in general position.

Refining our previous remarks, in order to investigate coherence problems, we need only consider reductions that are in general position in a 2-structure on a discrete category. In the following section, we tackle the problem of deciding whether such a diagram commutes.

3. LAMBEK COHERENCE

Given a 2-structure s on a category \mathcal{C} , we often wish to determine whether a diagram in $\mathbb{F}_{\mathcal{C}}(s)$ commutes. Such a diagram may commute due to commutativity of diagrams already present in \mathcal{C} , or it may commute purely as a result of the structure present in s . It is the latter case that concerns us here and, as such, we may make the assumption that \mathcal{C} is discrete.

Definition 3.1 (Lambek Coherence). A 2-structure on a category \mathcal{C} is Lambek coherent if it is decidable whether two reductions in general position having the same source and target are equal whenever \mathcal{C} is a discrete category.

An immediate question that arises is whether every 2-structure is Lambek coherent. Unsurprisingly, the answer is no, even in the case that the 2-structure is finitely presented.

Theorem 3.2. There exist finitely presented 2-structures that are not Lambek-coherent.

Proof. Let $\langle X|R \rangle$ be a finite presentation for a monoid with an unsolvable word problem. Let s be the structure consisting of a single unary function symbol F , reductions $\tau_i : F(x) \rightarrow F(x)$ for every $\tau_i \in X$ and relations (ω_i, ω_j) for every $(\omega_i, \omega_j) \in R$. Then the Lambek coherence problem for s is equivalent to the word problem for $\langle X|R \rangle$ and is hence undecidable. \square

Seeking to understand the reasons why a 2-structure could fail to be Lambek coherent, one may well suspect that termination is a key feature.

Definition 3.3 (Termination). *A 2-structure on a category \mathcal{C} is terminating if whenever \mathcal{C} is a discrete category, every infinite chain*

$$t_1 \xrightarrow{\alpha_1} t_2 \xrightarrow{\alpha_2} t_3 \xrightarrow{\alpha_3} \dots$$

in $\mathbb{F}_{\mathcal{C}}(s)$ contains cofinitely many identity reductions.

One may reasonably put forward the question as to whether every *terminating* 2-structure is Lambek coherent. It is a classical result of term rewriting theory that termination is an undecidable property (see, for example, [7]). Since the example constructed in Theorem 3.2 is not terminating, it is entirely possible that this is the point at which undecidability of Lambek coherence creeps in. In this section, we show that this intuition is roughly correct. In fact, we require a slightly weaker property than termination, which allows the result to be applicable to systems such as that consisting solely of a unary function symbol F and the reduction rule $F(x) \rightarrow F(F(x))$. However, we do need to work modulo the decidability of the word problem at the object level.

Our general approach is to examine the collection of subdivisions of a given parallel pair of reductions in general position. We seek a general criterion that ensures that any such pair admits only finitely many subdivisions. If this is the case, we may enumerate the subdivisions of a given parallel pair and examine each face for commutativity. We first need to develop an appropriate definition of a subdivision.

3.1. Subdivisions. A subdivision of a parallel pair of reductions is, in the first instance, a collection of reductions having the same source and target.

Definition 3.4. *An st-graph is a labelled directed graph G (possibly with loops and multiple edges) together with two distinguished vertices u and v , called the source and target of G respectively, such that for any other vertex $w \in G$, there exist paths $u \rightarrow w$ and $w \rightarrow v$ in G .*

Of particular interest to us are st-graphs contained in the reduction graph of a 2-structure.

Definition 3.5. *A morphism $\varphi \in \mathbb{F}_{\mathcal{C}}(\mathcal{M})$ is irreducible if $\varphi = \varphi_1 \cdot \varphi_2$ implies that $\varphi_1 = 1$ or $\varphi_2 = 1$.*

Definition 3.6 (Reduction graph). *Let s be a 2-structure on a discrete category \mathcal{C} . The expression $\text{Red}_{s, \mathcal{C}}$ denotes the reduction graph of s on \mathcal{C} . This graph has*

- *Vertices:* The set $\text{Term}_{\text{Ob}(\mathcal{C})}(\mathcal{F}, \theta_{\mathcal{F}})$.
- *Edges:* Irreducible morphisms in $\mathbb{F}_{\mathcal{C}}(s)$.

A subdivision corresponds to a particular way of embedding an st-graph in the oriented plane. Given a graph G , we use $|G|$ to denote its geometric realisation. We write \mathbb{R}^2 for the plane with the clockwise orientation.

Definition 3.7. Let G be a graph and $\alpha, \beta \in G(s, t)$. A pre-subdivision of $\langle \alpha, \beta \rangle$ is a pair (S, φ) such that:

- (1) S is an st-graph.
- (2) $\{\alpha, \beta\} \subseteq S \subseteq G$.
- (3) $\varphi : |S| \hookrightarrow \mathbb{R}^2$ is a planar embedding.
- (4) For every edge $\gamma \in S$, the image $\varphi(|\gamma|)$ is contained in the region of \mathbb{R}^2 bounded by $\varphi(|\alpha|)$ and $\varphi(|\beta|)$.

We use $\text{PSub}_G(\alpha, \beta)$ to denote the set of all pre-subdivisions of $\langle \alpha, \beta \rangle$ in G .

The definition of pre-subdivisions admits too many different embeddings of the same graph. To this end, we define a useful equivalence relation on pre-subdivisions.

Given a graph G , let $\alpha, \beta \in G(s, t)$. Let $\langle S_1, \varphi \rangle$ and $\langle S_2, \psi \rangle$ be pre-subdivisions of $\langle \alpha, \beta \rangle$. Define \sim to be the equivalence relation on $\text{PSub}_G(\alpha, \beta)$ generated by setting $\langle S_1, \varphi \rangle \sim \langle S_2, \psi \rangle$ if:

- (1) $S_1 = S_2$.
- (2) φ and ψ are ambiently isotopic.

We write $\text{Sub}_G(s, t)$ for the quotient $\text{PSub}_G(s, t) / \sim$.

Definition 3.8. For a directed graph G and $\alpha, \beta \in G(s, t)$, a subdivision of $\langle \alpha, \beta \rangle$ is a member of $\text{Sub}_G(s, t)$. For a 2-structure s on a discrete category \mathcal{C} , a subdivision of a parallel pair of morphisms $\alpha, \beta \in \mathbb{F}_{\mathcal{C}}(S)$ is a subdivision of $\langle \alpha, \beta \rangle$ in $\text{Red}_{s, \mathcal{C}}$. The set of all such subdivisions is denoted $\text{Sub}_{s, \mathcal{C}}(\alpha, \beta)$.

Recall that a directed graph G is *locally finite* if $G(s, t)$ is finite for all vertices $s, t \in G$. The following sequence of lemmas establishes a correspondence between local finiteness and finitely many subdivisions.

Lemma 3.9. For a directed graph G and a finite planar subgraph $S \leq G(s, t)$, there are only finitely many subdivisions of $\alpha, \beta \in G(s, t)$ having graph S .

Proof. Since we only consider embeddings of S up to ambient isotopy, a subdivision with graph S is completely determined by the set of edges mapped to the region bounded by $\varphi(|\gamma_1|)$ and $\varphi(|\gamma_2|)$ for every parallel pair of paths $\gamma_1, \gamma_2 \in S$. Since S is finite, there are only finitely many possibilities for this. \square

Lemma 3.10. An st-graph with source s and target t is finite if and only if it has finitely many planar st-subgraphs with source s and target t .

Proof. (\Rightarrow) A finite graph has finitely many subgraphs, so it certainly has finitely many planar subgraphs.

(\Leftarrow) Suppose that G is an infinite st-graph with source s and target t . Each path from s to t in G determines a planar subgraph of G , hence G has infinitely many planar subgraphs with source s and target t . \square

Combining the Lemma 3.9 and Lemma 3.10, we obtain the desired correspondence.

Lemma 3.11. If G is a directed graph containing vertices s and t , then $G(s, t)$ is finite if and only if $\text{Sub}_G(\alpha, \beta)$ is finite for all $\alpha, \beta \in G(s, t)$

3.2. Ensuring local finiteness. By Lemma 3.11, in order to ensure that every parallel pair of paths in a directed graph has finitely many subdivisions, we need only establish that the graph is locally finite. To this end, we make the following definition.

Definition 3.12. Let G be a directed graph. A quasicycle in G is a pair (T, t) such that:

- (1) T is an infinite chain $t_0 \rightarrow t_1 \rightarrow \dots$ in G
- (2) t is a vertex in G .
- (3) G contains a path $t_i \rightarrow t$ for all $i \in \mathbb{N}$.

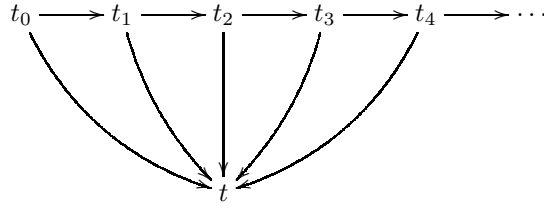


FIGURE 2. A quasicycle

Quasicycles earn their name by being a slightly weaker notion than a cycle. Figure 2 gives an example of a quasicycle that is not a cycle. On the other hand, we have the following easy result.

Lemma 3.13. Let C be a directed cycle and c be a vertex in C . Then, (C, c) is a quasicycle.

For a directed graph G and a vertex $s \in G$, we use $\text{Out}_G(s)$ to denote the set $\{t \in V(G) : G \text{ contains an edge } s \rightarrow t\}$. We say that G is *finitely branching* if $\text{Out}_G(s)$ is finite for all vertices $s \in G$. One of our main technical tools is the following graphical version of König's Tree Lemma.

Lemma 3.14. A finitely branching directed graph is locally finite if and only if it contains no quasicycles.

Proof. Let G be a labelled finitely branching directed graph.

(\Rightarrow) Suppose that G contains a quasicycle (T, t) , where $T = t_0 \xrightarrow{\alpha_0} t_1 \xrightarrow{\alpha_1} \dots$. If $t_i = t$ for some $i \in \mathbb{N}$ then $G(t_i, t_j)$ is infinite for all $j > i$. So, suppose that $t_i \neq t$ for all $i \in \mathbb{N}$. Since $t_i \rightarrow t$ for all $i \in \mathbb{N}$, there must be infinitely many pairs (i, β_i) , where $i \in \mathbb{N}$ and $\beta_i : t_i \rightarrow t$ is a path that does not factor through t_j for any $j > i$. So, $G(t_0, t)$ is infinite.

(\Leftarrow) Suppose that $G(s, t)$ is infinite. Since $\text{Out}_G(s)$ is finite, it follows from the pigeon hole principle that there must exist some $s_0 \in \text{Out}_G(s)$ and an edge $\alpha_0 : s \rightarrow s_0$ such that $G(s_0, t)$ is infinite. Continuing recursively, we obtain an infinite chain $s \xrightarrow{\alpha_0} s_0 \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} \dots$ such that G contains a path $s_i \rightarrow t$ for all $i \in \mathbb{N}$. So, G contains a quasicycle. \square

Definition 3.15. A 2-structure s on a category \mathcal{C} is quasicycle-free if every quasicycle in $\text{Red}_{s, \mathcal{C}}$ contains cofinitely many identity reductions. It is locally finite if $\text{Red}_{s, \mathcal{C}}$ is locally finite.

Lemma 3.14 very quickly yields the following fundamental result.

Proposition 3.16. *A finitely presented 2-structure on a discrete category is locally finite if and only if it is quasicycle-free.*

Proof. Let s be a finitely presented 2-structure on a discrete category \mathcal{C} . Since each term has finitely many subterms and s has finitely many reduction rules, $\text{Red}_{s,\mathcal{C}}$ is finitely branching. Lemma 3.14 then applies. \square

Lemma 3.11 and Proposition 3.16 together imply that a finitely presented quasicycle-free 2-structure on a discrete category has only finitely many subdivisions for every parallel pair of reductions. A ready supply of such 2-structures is provided by the following lemma.

Lemma 3.17. *A terminating 2-structure on a discrete category is quasicycle free.*

By Lemma 3.13, a quasicycle-free directed graph is acyclic. The following theorem establishes that every face of a subdivision in an acyclic graph is itself a parallel pair of paths. It was originally discovered by Power [13] in his investigation of pasting diagrams in 2-categories.

Theorem 3.18 (Power [13]). *A planar st-graph is acyclic if and only if every face has a unique source and target.*

Theorem 3.18 allows us to very easily deduce the following result.

Proposition 3.19. *Let s be a 2-structure on a discrete category \mathcal{C} and $\alpha, \beta \in \text{Red}_{s,\mathcal{C}}(s, t)$. Then, the following statements are equivalent:*

- (1) $\alpha = \beta$ in $\mathbb{F}_{\mathcal{C}}(s)$.
- (2) *There is a subdivision of $\langle \alpha, \beta \rangle$ in $\text{Red}_{s,\mathcal{C}}(s, t)$ such that each face commutes in $\mathbb{F}_{\mathcal{C}}(s)$.*
- (3) *There is a subdivision of $\langle \alpha, \beta \rangle$ in $\text{Red}_{s,\mathcal{C}}(s, t)$ such that each face is either an instance of functoriality, or an instance of naturality or an instance of one of the equations in $\theta_{\mathcal{T}}$.*

3.3. The Lambek coherence theorem. With Proposition 3.19, Proposition 3.16 and Lemma 3.11, we are seemingly home and dry since we now know that every quasicycle-free 2-structure on a discrete category has only finitely many subdivisions for each parallel pair and we can just check every face to see whether it is an instance of functoriality, naturality or a coherence axiom. There is, however, one catch - we may not be able to decide whether a given face is an instance of an axiom!

Definition 3.20 (Unification). *Let \mathcal{F} be a ranked set of function symbols on a set X and $\theta_{\mathcal{F}}$ be an equational theory on $\text{Term}_X(\mathcal{F})$. A $\theta_{\mathcal{F}}$ -unification problem is a finite set:*

$$\Gamma = \{(s_1, t_1), \dots, (s_n, t_n)\},$$

where for $1 \leq i \leq n$, we have that s_i and t_i are in $\text{Term}_{\mathcal{F}}(X)$. A unifier for Γ is a homomorphism $\sigma : \text{Term}_{\mathcal{F}}(X) \rightarrow \text{Term}_{\mathcal{F}}(X)$ such that $\sigma(s_i) =_{\theta_{\mathcal{F}}} \sigma(t_i)$ for all $1 \leq i \leq n$. The set Γ is unifiable if it admits at least one unifier.

Unification theory is an important technical component of automated reasoning and logic programming, as it provides a means for testing whether two sequences of terms are syntactic variants of each other. A good survey of the field is provided by [1]. In the case where the theory $\theta_{\mathcal{F}}$ is empty, the unification problem is easily

shown to be decidable. Unfortunately, the equational unification problem is in general undecidable.

Definition 3.21. *A 2-structure $\langle \mathcal{F}, \theta_{\mathcal{F}}, \mathcal{T}, \theta_{\mathcal{T}} \rangle$ has decidable term unification if $\langle \mathcal{F}, \theta_{\mathcal{F}} \rangle$ has a decidable unification problem.*

We can finally establish the main theorem of this section.

Theorem 3.22 (Lambek Coherence). *A finitely presented quasicycle-free structure with decidable term unification on a discrete category is Lambek Coherent.*

Proof. Let s be a 2-structure on a discrete category \mathcal{C} satisfying the hypotheses and $\alpha, \beta \in \text{Red}_s(s, t)$. By Proposition 3.16 and Lemma 3.11, we can enumerate the subdivisions of $\langle \alpha, \beta \rangle$. Since each subdivision has only finitely many faces, we may apply Proposition 3.19 to determine whether every face of a subdivision commutes in $\mathbb{F}_{\mathcal{C}}(s)$, since s has decidable term unification. \square

Corollary 3.23. *It is undecidable whether a finitely presented discrete structure with decidable term unification is quasicycle-free.*

Proof. The discrete structure constructed in the proof of Theorem 3.2 clearly has an empty equational theory on terms and so has decidable term unification. It follows from Theorem 3.22 that, were we able to determine whether the structure is quasicycle free, then we would be able to decide whether a finite monoid presentation has a decidable word problem. \square

As a particular application of Theorem 3.22, any terminating 2-structure with an empty equational theory is Lambek coherent. This includes, amongst others, categories with a directed associativity [9]. The unification problem for an associative binary symbol is decidable [1]. It follows then, from Theorem 3.22 that the following 2-structures are Lambek coherent (in each case we need only check that the 2-structure is terminating):

- Distributive categories with strict associativities and strict units [10]
- Weakly distributive categories with strict associativity and strict units [3].

An example of a non-terminating 2-structure that is Lambek-coherent is provided by the system $\mathcal{F}(X) \rightarrow F(F(X))$, since this is easily seen to be quasicycle free.

In the following section, we continue our investigation of quasicycle free 2-structures and derive sufficient conditions for such a system to be Mac Lane coherent.

4. MAC LANE COHERENCE

The last section was concerned with deciding whether a given pair of parallel morphisms is equal or, equivalently, whether a given diagram in general position commutes. In this section, we tackle the problem of determining sufficient conditions for all such diagrams to commute.

Definition 4.1. *Let s be a 2-structure on a discrete category \mathcal{C} . We say that s is Mac Lane coherent if every pair of morphisms in general position in $\mathbb{F}_{\mathcal{C}}(s)$ with the same source and target are equal.*

Our rough goal in this section is to find a minimal set of diagrams in general position whose commutativity implies the commutativity of all other such diagrams in $\mathbb{F}_{\mathcal{C}}(s)$ for some 2-structure s on a discrete category \mathcal{C} . To this end, we define

what it means for one subdivision to be finer than another. The idea driving idea is that we only wish to consider those subdivisions that do not embed into a finer subdivision.

Definition 4.2. Let s be a 2-structure on a discrete category \mathcal{C} , and $\alpha, \beta \in \text{Red}_{s, \mathcal{C}}(s, t)$ and $(S_1, \varphi), (S_2, \psi) \in \text{Sub}_{s, \mathcal{C}}(\alpha, \beta)$. We say that (S_1, φ) is coarser than (S_2, ψ) if there is a graph embedding $\Lambda : S_1 \rightarrow S_2$ making the following diagram commute. In this case, we also say that (S_2, ψ) is finer than (S_1, φ) and we write $(S_1, \varphi) \preceq (S_2, \psi)$.

$$\begin{array}{ccc} S_1 & \xrightarrow{|\cdot|} & |S_1| \\ \Lambda \downarrow & & \downarrow |\Lambda| \\ S_2 & \xrightarrow{|\cdot|} & |S_2| \end{array} \quad \begin{array}{c} \xrightarrow{\varphi} \\ \mathbb{R}^2 \\ \xleftarrow{\psi} \end{array}$$

We define the refinement order to be the antisymmetric closure of \preceq .

We shall abuse notation slightly and henceforth write \preceq for the refinement order. It is immediate from the definitions that the set of subdivisions of a parallel pair of morphisms forms a poset under refinement.

Definition 4.3. Let s be a 2-structure on a discrete category \mathcal{C} and $\alpha, \beta \in \text{Red}_{s, \mathcal{C}}(s, t)$. A maximal subdivision of $\langle \alpha, \beta \rangle$ is a maximal element of $(\text{Sub}_{s, \mathcal{C}}(\alpha, \beta), \preceq)$.

The idea behind the definition of a maximal subdivision is that these are precisely the ones which cannot be further subdivided. Indeed, we have the following lemma.

Lemma 4.4. A finitely presented quasicycle free 2-structure on a discrete category is Mac Lane coherent if and only if every parallel pair of reductions in general position admits a maximal subdivision, each face of which commutes.

Proof. The direction (\Leftarrow) is trivial. For the other direction, let s be a quasicycle-free 2-structure on a discrete category \mathcal{C} . Let $\alpha, \beta \in \text{Red}_{s, \mathcal{C}}(s, t)$. Since s is quasicycle-free, it follows from Proposition 3.16 and Lemma 3.11 that $\text{Sub}_{s, \mathcal{C}}(\alpha, \beta)$ is finite. Therefore, $\langle \alpha, \beta \rangle$ admits a maximal subdivision (S, φ) . By Theorem 3.18, every face of (S, φ) has a unique source and target. Since s is Mac Lane coherent, each of these faces commutes. \square

In order to make Lemma 4.4 effective, we need to characterise those parallel pairs of morphisms that can occur as faces of a maximal subdivision.

Definition 4.5 (Zig-zag subdivision). Let G be a directed graph and $\alpha, \beta \in G(s, t)$. Suppose that

$$\begin{aligned} \alpha &= s \xrightarrow{\alpha_0} a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} a_{n-1} \xrightarrow{\alpha_n} t \\ \beta &= s \xrightarrow{\beta_0} b_0 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{m-1}} b_{m-1} \xrightarrow{\beta_m} t \end{aligned}$$

and that each α_i and β_i is irreducible. Let U be the forgetful functor from directed graphs to graphs that forgets the direction of edges. A zig-zag subdivision of $\langle \alpha, \beta \rangle$ is a subdivision (S, φ) of $\langle \alpha, \beta \rangle$ such that $U(S)$ contains a path from $U(a_i)$ to $U(b_j)$ for some pair (i, j) , with $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$. We call the preimage of this path the zig-zag of S .

Definition 4.6 (Diamond). Let G be a directed graph. A pair $\alpha, \beta \in G(s, t)$ is called a diamond if it does not admit a zig-zag subdivision.

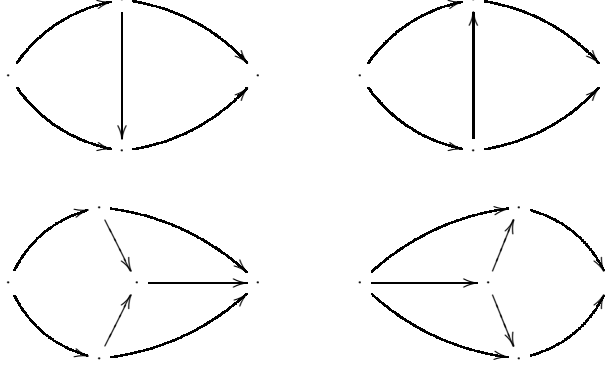


FIGURE 3. A few zig-zag subdivisions.

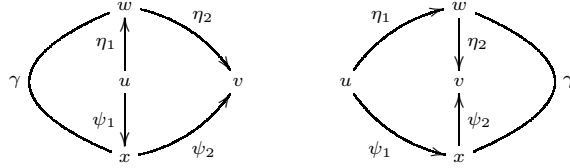
The idea behind the definition of a diamond is that any subdivision containing a face that admits a zig-zag subdivision cannot be a maximal subdivision. This is made precise in the following proposition.

Proposition 4.7. *Let G be an acyclic directed graph and $\alpha, \beta \in G(s, t)$. Every face of a maximal subdivision of $\langle \alpha, \beta \rangle$ is a diamond.*

Proof. Let G be an acyclic directed graph and let (S, φ) be a maximal subdivision of $\alpha, \beta \in G(s, t)$. By Theorem 3.18, every face of S has a unique source and target. That is, every face consists of a parallel pair of reductions $\eta, \psi : u \rightarrow v$. Suppose that $\langle \eta, \psi \rangle$ is a face of S that is not a diamond. That is, it admits a zig-zag subdivision. So, we have

$$\begin{aligned} \eta &= u \xrightarrow{\eta_1} w \xrightarrow{\eta_2} v \\ \psi &= u \xrightarrow{\psi_1} x \xrightarrow{\psi_2} v, \end{aligned}$$

and a zig-zag γ between w and x that is a part of a subdivision of $\langle \eta, \psi \rangle$. By maximality, γ must be contained in S . Since $\langle \eta, \psi \rangle$ is a face, $\varphi(|\gamma|)$ cannot lie in the region bounded by $\varphi(|\eta|)$ and $\varphi(|\psi|)$. So, we are in one of the situations depicted in Figure 4.

FIGURE 4. Possible embeddings of γ .

Suppose that we are in the situation depicted in the left hand diagram of Figure 4. Since γ is contained in S and since S is an st-graph, there is a path $s \xrightarrow{\delta} u$. By planarity, δ must factor through a vertex in γ or η_2 or ψ_2 . If δ factors through a vertex in η_2 or ψ_2 then it is clear that G contains a cycle, contradicting the fact that G is acyclic. So, we must have $s \xrightarrow{\delta_1} z \xrightarrow{\delta_2} u$ for some vertex z in γ . However, since γ

appears in a subdivision of $\langle \eta, \psi \rangle$, there is a path $u \xrightarrow{\zeta} z$ in G . Then, $\delta_2 \cdot \zeta$ forms a cycle in G , contradicting the fact that G is acyclic. So, γ can not be embedded as in the left hand picture of Figure 4. Dually, it cannot be embedded as in the right hand picture of Figure 4.

Therefore, $\langle \eta, \psi \rangle$ admits a zig-zag subdivision with zig-zag γ , contradicting the maximality of (S, φ) . So, $\langle \eta, \psi \rangle$ must be a diamond. \square

Combining Lemma 4.4 and Proposition 4.7, we obtain our general Mac Lane Coherence theorem.

Theorem 4.8. *[Mac Lane Coherence] A finitely presented quasicycle-free structure s on a discrete category \mathcal{C} is Mac Lane coherent if and only if every diamond in $\text{Red}_{s, \mathcal{C}}$ commutes in $\mathbb{F}_{\mathcal{C}}(s)$.* \square

Theorem 4.8 says that in order to show that a 2-structure on a discrete category is Mac Lane coherent, we need to do two things:

- (1) Show that $\mathbb{F}_{\mathcal{C}}(s)$ is quasicycle-free.
- (2) Show that every diamond commutes.

At the onset, showing that every diamond commutes can be a daunting task. We can guide our investigations by exploiting some standard term rewriting theory [6].

Definition 4.9. *Let s be a 2-structure on a category \mathcal{C} and let φ_1 and φ_2 be irreducible morphisms in $\mathbb{F}_{\mathcal{C}}(s)$. We call $\langle \varphi, \psi \rangle$ the initial span in a diagram of the following form:*

$$\begin{array}{ccc} & \xrightarrow{\varphi_1} & \\ \psi_1 \downarrow & & \downarrow \varphi_2 \\ & \xrightarrow{\psi_2} & \end{array}$$

If φ_1 and ψ_1 are irreducible, then there are three possibilities for a diamond with initial span $\langle \varphi, \psi \rangle$:

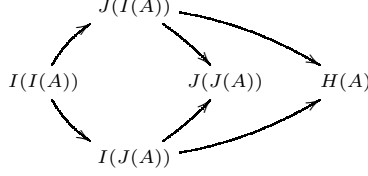
- (1) φ and ψ rewrite disjoint subterms.
- (2) φ and ψ rewrite nested subterms.
- (3) φ and ψ rewrite overlapping subterms. Without loss of generality, we may assume that $\langle \varphi, \psi \rangle$ forms a critical peak.

By analogy with the critical pairs lemma [6], one may hope to reduce the problem to only examining diamonds whose initial span is a critical pair. Unfortunately, as the following two examples show, there may be more than one diamond whose initial span performs a given pair of nested or disjoint rewrites.

Example 4.10. *In this example we construct a terminating 2-structure that has more than one diamond with the same initial span performing a nested rewrite. Let s be the 2-structure consisting of unary functor symbols I, J and H , together with the following reduction rules:*

$$\begin{aligned} I(x) &\rightarrow J(x) \\ I(J(x)) &\rightarrow H(x) \\ J(I(x)) &\rightarrow H(x) \end{aligned}$$

Let \mathcal{C} be the discrete category generated by $\{A\}$. Then, $\mathbb{F}_{\mathcal{C}}(s)$ contains the following diagram:

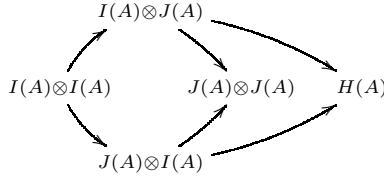


Since there is no reduction $J(J(A)) \rightarrow H(A)$, both parallel reductions form diamonds.

Example 4.11. In this example we construct a terminating 2-structure that has more than one diamond with the same initial span performing a disjoint rewrite. Let s be the 2-structure consisting of unary functor symbols I and J , the binary functor symbol \otimes and the following reduction rules:

$$\begin{aligned} I(x) &\rightarrow J(x) \\ J(x) \otimes I(x) &\rightarrow H(x) \\ I(x) \otimes J(x) &\rightarrow H(x) \end{aligned}$$

Let \mathcal{C} be the discrete category generated by $\{A\}$. Then, $\mathbb{F}_{\mathcal{C}}(s)$ contains the following diagram:



Since there is no reduction $J(A) \otimes J(A) \rightarrow H(A)$, both parallel reductions form diamonds.

Examples 4.10 and 4.11 serve to warn us that the collection of diamonds behaves a lot more subtly than the collection of spans, which are the typical objects of study in traditional term rewriting theory. Before illustrating the next subtle point about quasicycle-free 2-structures, we separate those that are inherently infinite from those that are inherently finite.

Definition 4.12 ((Finitely) coherently axiomatisable). Let $\mathcal{R} := \langle \mathcal{F}, \theta_{\mathcal{F}}, T \rangle$ be a term rewriting theory. We say that \mathcal{R} is coherently axiomatisable if there is a set of equations, θ_T , between reductions having the same source and target such that $\langle \mathcal{F}, \theta_{\mathcal{F}}, T, \theta_T \rangle$ is a Mac Lane coherent 2-structure. We say that \mathcal{R} is finitely coherently axiomatisable if it is finitely presented and there is a finite such θ_T .

Theorem 4.8 immediately yields the following:

Theorem 4.13. A quasicycle-free 2-structure is coherently axiomatisable.

Proof. Add all diamonds as axioms and apply Theorem 4.8. \square

Since quasicycle-freeness was enough to guarantee only finitely many subdivisions of a given parallel pair, one may hope that every finitely presented quasicycle-free 2-structure is finitely coherently axiomatisable. Sadly, this is not the case.

Proposition 4.14. *There exist finitely presented coherently axiomatisable 2-structures that are not finitely coherently axiomatisable.*

Proof. Let s be the 2-structure containing unary functor symbols F, G, I and H , together with the following reduction rules:

$$\begin{aligned} I(x) &\rightarrow G(I(x)) \\ I(x) &\rightarrow F(I(x)) \\ F(x) &\rightarrow F(F(x)) \\ G(x) &\rightarrow G(G(x)) \\ F(x) &\rightarrow H(x) \\ G(x) &\rightarrow H(x) \end{aligned}$$

Let \mathcal{C} be the discrete category generated by $\{A\}$. It is clear that $\mathbb{F}_{\mathcal{C}}(s)$ is quasicycle-free, so taking all diamonds as axioms, Theorem 4.8 implies that s is coherently axiomatisable. However, $\mathbb{F}_{\mathcal{C}}(s)$ contains the following diagram:

$$\begin{array}{ccccccc} & & G(I(A)) & \longrightarrow & G^2(I(A)) & \longrightarrow & G^3(I(A)) & \longrightarrow & \cdots \\ & \nearrow & \downarrow & & \downarrow & & \downarrow & & \\ I(A) & & H(I(A)) & & H^2(I(A)) & & H^3(I(A)) & & \cdots \\ & \searrow & \uparrow & & \uparrow & & \uparrow & & \\ & & F(I(A)) & \longrightarrow & F^2(I(A)) & \longrightarrow & F^3(I(A)) & \longrightarrow & \cdots \end{array}$$

Since there are no reductions $H^i(A) \rightarrow H^j(A)$ for $i \neq j$, no finite collection of diamonds with source $I(A)$ implies the commutativity of all others. So, s is not finitely coherently axiomatisable. \square

In this section, we have derived a very general Mac Lane coherence theorem and used it to illuminate some of the many subtleties of coherence problems for covariant structures. In the following section, we apply this theory to a substantial coherence problem.

5. COHERENCE FOR ITERATED MONOIDAL CATEGORIES

Iterated monoidal categories were introduced in [2], in order to make precise the intuition that the category of monoids internal to a category corresponds to the space of loops internal to a topological space. Iterating the construction of internal monoids, one arrives at the concept of an n -fold monoidal category. The basic structure of [2] is to unpack the definition in terms of internal monoids in order to obtain a categorical operad characterising n -fold monoidal categories and to subsequently derive a weak homotopy equivalence between the nerve of this operad and the little n -cubes operad.

The presentation of the operadic theory for iterated monoidal categories in [2] utilises strict associativity and unit maps. Thus, there is a nontrivial congruence present at both the object level and the structure level. This two-level structure leads to a subtle interplay between the object-level equational theory and the reductions. The coherence problem is further compounded by the fact that n -fold

monoidal categories do not have unique normal forms. A coherence theorem is obtained in [2], which says that there is a unique map in an n -fold monoidal category between two terms without repeated variables. The proof proceeds via an intricate double induction on the number of variables and the dimension of the outermost tensor product in the target of a morphism. In this section, we exploit Theorem 4.8 to provide a new, conceptual proof of the coherence theorem for iterated monoidal categories.

Definition 5.1. *The 2-structure for n -fold monoidal categories is denoted \mathcal{M}_n and consists of the following.*

- (1) n binary functor symbols:

$$\otimes_1, \dots, \otimes_n : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

- (2) A nullary functor symbol I

- (3) For $1 \leq i \leq n$:

$$A \otimes_i (B \otimes_i C) = (A \otimes_i B) \otimes_i C$$

$$A \otimes_i I = A$$

$$I \otimes_I A = A$$

- (4) For each pair (i, j) such that $1 \leq i < j \leq n$, there is a reduction rule, called *interchange*:

$$\eta_{A,B,C,D}^{ij} : (A \otimes_j B) \otimes_i (C \otimes_j D) \rightarrow (A \otimes_i C) \otimes_j (B \otimes_i D)$$

The interchange rules are subject to the following conditions:

- (1) *Internal unit Condition:* $\eta_{A,B,I,I}^{ij} = \eta_{I,I,A,B}^{ij} = id_{A \otimes_j B}$.
(2) *External unit condition:* $\eta_{A,I,B,I}^{ij} = \eta_{I,A,I,B}^{ij} = id_{A \otimes_i B}$.
(3) *Internal associativity condition.* The following diagram commutes:

$$\begin{array}{ccc} (A \otimes_j B) \otimes_i (C \otimes_j D) \otimes_i (E \otimes_j F) & \xrightarrow{\eta_{A,B,C,D}^{ij} \otimes_i id_{E \otimes_j F}} & ((A \otimes_i C) \otimes_j (B \otimes_i D)) \otimes_i (E \otimes_j F) \\ \downarrow id_{A \otimes_j B} \otimes_i \eta_{C,D,E,F}^{ij} & & \downarrow \eta_{A \otimes_i C, B \otimes_i D, E, F}^{ij} \\ (A \otimes_j B) \otimes_i ((C \otimes_i E) \otimes_j (D \otimes_i F)) & \xrightarrow{\eta_{A,B,C \otimes_i E, D \otimes_i F}^{ij}} & (A \otimes_i C \otimes_i E) \otimes_j (B \otimes_i D \otimes_i F) \end{array}$$

- (4) *External associativity condition.* The following diagram commutes:

$$\begin{array}{ccc} (A \otimes_j B \otimes_j C) \otimes_i (D \otimes_j E \otimes_j F) & \xrightarrow{\eta_{A \otimes_j B, C, D \otimes_j E, F}^{ij}} & ((A \otimes_j B) \otimes_i (D \otimes_j C)) \otimes_j (C \otimes_i F) \\ \downarrow \eta_{A,B \otimes_j C, D, E \otimes_j F}^{ij} & & \downarrow \eta_{A,B,D,C \otimes_j id_{C \otimes_i F}}^{ij} \\ (A \otimes_i D) \otimes_j ((B \otimes_j C) \otimes_i (E \otimes_j F)) & \xrightarrow{id_{A \otimes_i D} \otimes_j \eta_{B,C,E,F}^{ij}} & (A \otimes_i D) \otimes_j (B \otimes_i E) \otimes_j (C \otimes_i F) \end{array}$$

(5) *Giant hexagon condition. The following diagram commutes:*

$$\begin{array}{ccc}
 ((A \otimes_k B) \otimes_j (C \otimes_k D)) \otimes_i ((E \otimes_k F) \otimes_j (G \otimes_k H)) & & \\
 \swarrow \eta^{jk} \otimes_k \eta^{jk} & & \searrow \eta^{ij} \\
 ((A \otimes_j C) \otimes_k (B \otimes_j D)) \otimes_i ((E \otimes_j G) \otimes_k (F \otimes_j H)) & & ((A \otimes_k B) \otimes_i (E \otimes_k F)) \otimes_j ((C \otimes_k D) \otimes_i (G \otimes_k H)) \\
 \downarrow \eta^{ik} & & \downarrow \eta^{ik} \otimes_j \eta^{ik} \\
 ((A \otimes_j C) \otimes_i (E \otimes_j G)) \otimes_k ((B \otimes_j D) \otimes_i (F \otimes_j H)) & & ((A \otimes_i E) \otimes_k (B \otimes_i F)) \otimes_j ((C \otimes_i G) \otimes_k (D \otimes_i H)) \\
 \searrow \eta^{ij} \otimes_k \eta^{ij} & & \swarrow \eta^{jk} \\
 ((A \otimes_i E) \otimes_j (C \otimes_i G)) \otimes_k ((B \otimes_i F) \otimes_j (D \otimes_i H)) & &
 \end{array}$$

In the giant hexagon, (i, j, k) is such that $1 \leq i < j < k \leq n$ and the natural transformations have the evident components.

It is very easy to characterise those reductions in \mathcal{M}_n that are general position.

Lemma 5.2. *Let \mathcal{C} be a discrete category. A reduction $s \rightarrow t$ in $\mathbb{F}_{\mathcal{C}}(\mathcal{M}_n)$ is in general position if and only if s and t contain no repeated variables. \square*

Because of the fact that an n -fold monoidal category is strictly associative and has a strict unit, we can derive various useful maps via Eckmann-Hilton style arguments. Several of these maps will be of particular use to us. In the following, we assume that (i, j) is such that $1 \leq i < j \leq n$. The derived maps are as follows:

- (1) Dimension raising: $A \otimes_i B \xrightarrow{\iota_{A,B}^{ij}} A \otimes_j B$. This represents the following composition:

$$A \otimes_i B \xrightarrow{=} (A \otimes_j I) \otimes_i (I \otimes_j B) \xrightarrow{\eta^{ij}} (A \otimes_i I) \otimes_j (I \otimes_i B) \xrightarrow{=} A \otimes_j B$$

- (2) Twisted dimension raising: $A \otimes_i B \xrightarrow{\tau_{A,B}^{ij}} B \otimes_j A$. This represents the following composition:

$$A \otimes_i B \xrightarrow{=} (I \otimes_j A) \otimes_i (I \otimes_j B) \xrightarrow{\eta^{ij}} (I \otimes_i B) \otimes_j (A \otimes_i I) \xrightarrow{=} B \otimes_j A$$

- (3) Left weak distributivity (This name is chosen to reflect the connection with weakly distributive categories [3]): $A \otimes_i (B \otimes_j C) \xrightarrow{\delta_{A,B,C}^{ij}} (A \otimes_i B) \otimes_j C$. This represents the following composition:

$$A \otimes_i (B \otimes_j C) \xrightarrow{=} (A \otimes_j I) \otimes_i (B \otimes_j C) \xrightarrow{\eta^{ij}} (A \otimes_i B) \otimes_j (I \otimes_i C) \xrightarrow{=} (A \otimes_i B) \otimes_j C$$

- (4) Twisted left weak distributivity: $A \otimes_i (B \otimes_j C) \xrightarrow{\tilde{\delta}_{A,B,C}^{ij}} B \otimes_j (A \otimes_i C)$. This represents the following composition:

$$A \otimes_i (B \otimes_j C) \xrightarrow{=} (I \otimes_j A) \otimes_i (B \otimes_j C) \xrightarrow{\eta^{ij}} (I \otimes_i B) \otimes_j (A \otimes_i C) \xrightarrow{=} B \otimes_j (A \otimes_i C)$$

- (5) Right weak distributivity: $(A \otimes_j B) \otimes_i C \xrightarrow{\gamma_{A,B,C}^{ij}} A \otimes_j (B \otimes_i C)$. This represents the following composition:

$$(A \otimes_j B) \otimes_i C \xrightarrow{=} (A \otimes_j B) \otimes_i (I \otimes_j C) \xrightarrow{\eta^{ij}} (A \otimes_i I) \otimes_j (B \otimes_i C) \xrightarrow{=} A \otimes_j (B \otimes_i C)$$

- (6) Twisted right weak distributivity: $(A \otimes_i B) \otimes_j C \xrightarrow{\tilde{\gamma}_{A,B,C}^{ij}} (A \otimes_j C) \otimes_i B$. This represents the following composition:

$$(A \otimes_i B) \otimes_j C \xrightarrow{=} (A \otimes_i B) \otimes_j (C \otimes_i I) \xrightarrow{\eta^{ij}} (A \otimes_j C) \otimes_i (B \otimes_j I) \xrightarrow{=} (A \otimes_j C) \otimes_i B$$

With the above maps, it is easy to see that iterated monoidal categories do not have unique normal forms.

Lemma 5.3. *Let \mathcal{C} be a discrete category. Then, $\mathbb{F}_{\mathcal{C}}(\mathcal{M}_n)$ is not confluent.*

Proof. The following span is clearly not joinable:

$$\begin{array}{ccc} A \otimes_i B & \xrightarrow{\iota_{A,B}^{in}} & A \otimes_n B \\ \tau_{A,B}^{in} \downarrow & & \\ B \otimes_n A & & \end{array}$$

□

Our first step is to bring iterated monoidal categories into the realm of applicability of Theorem 4.8.

Proposition 5.4. *\mathcal{M}_n is terminating.*

Proof. Let \mathcal{C} be a discrete category. We shall construct a ranking function on $\mathbb{F}_{\mathcal{C}}(\mathcal{M}_n)$. Define $\hat{\rho} : \text{Term}_{\text{Ob}\mathcal{C}}(\mathcal{M}_n) \rightarrow \mathbb{N}$ by:

$$\hat{\rho}(t) = \begin{cases} i + \hat{\rho}(A) + 2\hat{\rho}(B) & \text{if } t = A \otimes_i B \\ 0 & \text{if } t = I \end{cases}$$

At the moment, $\hat{\rho}$ is not a ranking function, since it is sensitive to the order of parenthesisation and the presence of units. We can, however, use it to construct a ranking function. Let $[t]$ be an object in $\mathbb{F}_{\mathcal{C}}(\mathcal{M}_n)$. Define:

$$\rho([t]) = \min\{\hat{\rho}(t') : t' \in [t]\}$$

The map ρ effectively calculates the rank of the member of a congruence class $[t]$ which has no units and the left most bracketing. It is immediate from the definition that, for $t_1, t_2 \in [t]$, we have $\rho(t_1) = \rho(t_2)$. We now need to show that interchange reduces the rank and must be careful to check the maps arising from Eckmann-Hilton arguments also. Let $j > i$.

- Interchange:

$$\rho((A \otimes_j B) \otimes_i (C \otimes_j D)) = i + 2j + \rho(A) + \rho(B) + \rho(C) + \rho(D)$$

$$\rho((A \otimes_i C) \otimes_j (B \otimes_i C)) = 2i + j + \rho(A) + \rho(B) + \rho(C) + \rho(D)$$

Since $j > i$, we have $i + 2j > 2i + j$.

- Left linear distributivity:

$$\rho(A \otimes_i (B \otimes_j C)) = i + 2j + \rho(A) + 2\rho(B) + 4\rho(C)$$

$$\rho((A \otimes_i B) \otimes_j C) = i + j + \rho(A) + 2\rho(B) + 2\rho(C)$$

The other cases are handled similarly. It follows that ρ is a ranking function on $\mathbb{F}_{\mathcal{C}}(\mathcal{M}_n)$, so $\mathbb{F}_{\mathcal{C}}(\mathcal{M}_n)$ is terminating. \square

It follows from Proposition 5.4 and Theorem 4.13 that \mathcal{M}_n is coherently axiomatisable. At this stage, however, we don't even know whether it is finitely coherently axiomatisable. Before examining diamonds in \mathcal{M}_n for commutativity, we recall some useful terminology and results from [2].

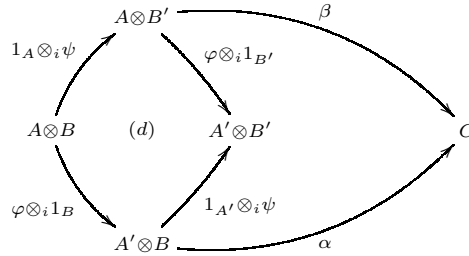
Let A be an object in $\mathbb{F}_{\mathcal{C}}(\mathcal{M}_n)$. For a set $X \subseteq \text{Var}(A)$, we write $A - X$ to denote the object resulting from substituting I for each variable in X . For instance $(A \otimes_i B) \otimes_j (C \otimes_i E) - \{B, D\} = A \otimes_j C$. We say that a term B is *in* a term A and write $B \in A$ if there is some $X \subseteq \text{Var}(A)$ such that $A - X = B$. Of crucial importance to us is the following result of [2].

Theorem 5.5 ([2]). *Let \mathcal{C} be a discrete category and let A and B be objects of $\mathbb{F}_{\mathcal{C}}(\mathcal{M}_n)$. A necessary and sufficient condition for the existence of a map $A \rightarrow B$ in $\mathbb{F}_{\mathcal{C}}(\mathcal{M})$ is that, for each $a, b \in \text{Var}(A)$, if $a \otimes_i b \in A$, then one of the following holds:*

- *There is some $j \geq i$ such that $a \otimes_j b \in B$*
- *There is some $j > i$ such that $b \otimes_j a \in B$*

Theorem 5.5 gives us the technical tool that we need in order to show that various parallel pairs of maps are not diamonds. We begin our analysis of the collection of diamonds of $\mathbb{F}_{\mathcal{C}}(\mathcal{M}_n)$ with diamonds whose initial span rewrites disjoint pieces of a term.

Lemma 5.6. *Let $A \otimes_i B \in \mathbb{F}_{\mathcal{C}}(\mathcal{M}_n)$ and suppose that there are maps $\varphi : A \rightarrow A'$ and $\psi : B \rightarrow B'$. Then, in the following diagram, the square labelled (d) is a commutative diamond and there is a map $A' \otimes_i B' \rightarrow C$:*



Proof. The square labelled (d) commutes by functoriality and it is easy to see that it does not admit a zig-zag subdivision, so it is a diamond. The tricky part is showing the existence of a map $A' \otimes_i B' \rightarrow C$.

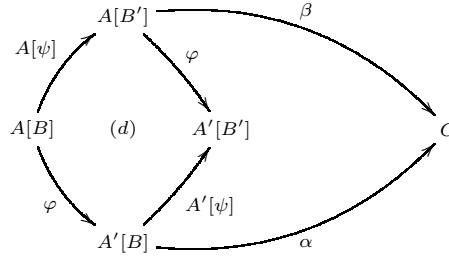
Let $X, Y \in \text{Var}(A' \otimes_i B')$ and suppose that $X \otimes_k Y \in A' \otimes_i B'$. There are a few cases to consider.

- If $X, Y \in A'$, then α implies that there is some $m \geq k$ such that $X \otimes_m Y \in C$ or there is some $m > k$ such that $Y \otimes_m X \in C$.
- If $X, Y \in B'$, then β implies that there is some $m \geq k$ such that $X \otimes_m Y \in C$ or there is some $m > k$ such that $Y \otimes_m X \in C$.
- If $X \in A'$ and $Y \in B'$, then $X \otimes_i Y \in A' \otimes B$. So, by α , there is some $m \geq i$ such that $X \otimes_m Y \in C$ or there is some $m > i$ such that $Y \otimes_m X \in C$.

Putting all of the above facts together, it follows from Theorem 5.5 that there is a map $A' \otimes_i B' \rightarrow C$. \square

Our next port of call is diamonds whose initial span rewrites nested subterms. For a term A and a subterm $B \leq A$, we write $A[B]$ to represent this nested term.

Lemma 5.7. *Let $A[B] \in \mathbb{F}_C(\mathcal{M}_n)$ and suppose that there are maps $\varphi : A[B] \rightarrow A'[B]$ and $\psi : B \rightarrow B'$. Then, in the following diagram, the square labelled (d) is a commutative diamond and there is a map $A'[B'] \rightarrow C$:*



Proof. The square labelled (d) commutes by naturality. The rest of the proof is similar to that of Lemma 5.6. \square

We now know that the only nontrivial diamonds in $\mathbb{F}_C(\mathcal{M}_n)$ have a critical pair as their initial span. Our remaining task is to perform a critical pairs analysis on \mathcal{M}_n .

5.1. Interchange + associativity. Let $j > i$. The first way in which interchange and associativity can interact is in the term $X \otimes_i (C \otimes_j D) \otimes_i (E \otimes_j F)$. Without loss of generality, we may assume that $X = A \otimes_j B$, because we could always take $X = X \otimes_j I$. The resulting span then gets completed into the internal associativity axiom. One may then apply Theorem 5.5 to show that there is no other diamond with the same initial span.

The second way in which interchange can interact with associativity is in the term $(A \otimes_j B) \otimes_i (C \otimes_j D \otimes_j E)$. In this case, we get the following square, where the labels have the evident components.

$$\begin{array}{ccc}
 (A \otimes_j B) \otimes_i (C \otimes_j D \otimes_j E) & \xrightarrow{\eta} & (A \otimes_i (C \otimes_j D)) \otimes_j (B \otimes_i E) \\
 \eta \downarrow & & \downarrow \delta \otimes_j 1 \\
 (A \otimes_i C) \otimes_j (B \otimes_i (D \otimes_j E)) & \xrightarrow{1 \otimes_j \delta} & (A \otimes_i C) \otimes_j D \otimes_j (B \otimes_i E)
 \end{array}$$

The above square commutes by substituting $(A \otimes_j I \otimes_j B) \otimes_i (C \otimes_j D \otimes_j E)$ for the source and using the external associativity axiom. Theorem 5.5 easily yields that there can be no other diamonds with the same initial span.

Similarly, a critical pair arises at $(A \otimes_j B \otimes_j C) \otimes_i (D \otimes_j E)$. The analysis is analogous to the previous case by inserting a unit to obtain $(A \otimes_j B \otimes_j C) \otimes_i (D \otimes_j I \otimes_j E)$.

5.2. Interchange + interchange. Let $i < j < k$. An overlap between interchange rules occurs at $(A \otimes_j B) \otimes_i ((C \otimes_k D) \otimes_j (E \otimes_j F))$. Since we have strict units, we may assume that $A = A_1 \otimes_k A_t$ and $B = B_1 \otimes_k B_2$. We then obtain the initial span of the giant hexagon axiom. The hexagon forms a diamond and it follows from Theorem 5.5 that there are no other diamonds with this initial span.

5.3. Interchange + units. The critical pairs arising from the interaction of interchange with units yield the various Eckmann-Hilton maps. As we have seen, these are not always joinable. When they are, they commute by the following lemma.

Lemma 5.8. *Let \mathcal{C} be a discrete category. The following diagrams commute in $\mathbb{F}_{\mathcal{C}}(\mathcal{M}_n)$, where $1 \leq i < j < k \leq n$:*

$$\begin{array}{ccc}
 & A \otimes_j B & \\
 \nearrow & (1) & \searrow \\
 A \otimes_i B & \xrightarrow{\quad} & A \otimes_k B
 \end{array}
 \qquad
 \begin{array}{ccc}
 & B \otimes_j A & \\
 \nearrow & (2) & \searrow \\
 A \otimes_i B & \xrightarrow{\quad} & A \otimes_k B
 \end{array}$$

$$\begin{array}{ccc}
 & A \otimes_j B & \\
 \nearrow & (3) & \searrow \\
 A \otimes_i B & \xrightarrow{\quad} & B \otimes_k A
 \end{array}
 \qquad
 \begin{array}{ccc}
 & B \otimes_j A & \\
 \nearrow & (4) & \searrow \\
 A \otimes_i B & \xrightarrow{\quad} & B \otimes_k A
 \end{array}$$

Proof. This follows from [2, Lemma 4.22]. More explicitly it follows from the giant hexagon axiom by making the following substitutions:

- (1) $A \otimes_i B = ((A \otimes_k I) \otimes_j (I \otimes_k I)) \otimes_i ((I \otimes_k I) \otimes_j (I \otimes_k B))$
- (2) $A \otimes_i B = ((I \otimes_k I) \otimes_j (A \otimes_k I)) \otimes_i ((I \otimes_k B) \otimes_j (I \otimes_k I))$
- (3) $A \otimes_i B = ((I \otimes_k A) \otimes_j (I \otimes_k I)) \otimes_i ((I \otimes_k I) \otimes_j (B \otimes_k I))$
- (4) $A \otimes_i B = ((I \otimes_k I) \otimes_j (I \otimes_k A)) \otimes_i ((B \otimes_k I) \otimes_j (I \otimes_k I))$

□

5.4. Putting it all together. We have seen that $\mathbb{F}_{\mathcal{C}}(\mathcal{M}_n)$ is terminating and that every diamond in $\mathbb{F}_{\mathcal{C}}(\mathcal{M}_n)$ commutes. We can therefore apply Theorem 4.8 to obtain the coherence theorem for iterated monoidal categories.

Theorem 5.9. *Let \mathcal{C} be a discrete category. If A and B are objects of $\mathbb{F}_{\mathcal{C}}(\mathcal{M}_n)$ having no repeated variables, then there is at most one map $A \rightarrow B$ in $\mathbb{F}_{\mathcal{C}}(\mathcal{M}_n)$. □*

6. CONCLUSIONS

Both of our general coherence theorems, Theorem 3.22 and Theorem 4.8, rely on the underlying structure being quasicycle-free. One may well call this condition into question and wonder whether we can get away with a weaker condition. For Lambek Coherence, quasicycle-freeness does not capture all covariant structures known to be Lambek coherent. For example, braided monoidal categories are certainly not quasicycle free and yet they are well known to be Lambek coherent. However, the method for proving this adds a rewrite system to the reductions, thus expanding the amount of information available. The question still stands, then, of whether

there is a property of the underlying term rewriting system that leads to Lambek coherence for non-quasicycle-free structures.

The reliance on quasicycle-freeness for Mac Lane coherence seems more fundamental. However, two of the crucial ingredients of our theory, Theorem 3.18 and Proposition 4.7 rely solely on acyclicity. This leads us to ask whether we can find conditions on an acyclic 2-structure that ensure Mac Lane coherence.

Nevertheless, our focus on quasicycle-free structures has proven to be broad and powerful enough for us to find the conceptual reason for the coherence theorem for iterated monoidal categories. Moreover, it has allowed us to show that there is a wide variety of coherence phenomena, even in the purely covariant case.

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